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REMARKS ON WAVE SOLUTIONS OF THE NONLINEAR HEAT-CONDUCTION EQUATION

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Wave solutions of the nonlinear heat-conduction equation are analyzed and their relation to self-similar solutions is established. Solutions of the hyperbolic and the nonlinear heat-conduction equations are compared.

1. Undamped Thermal Waves

Let us consider the nonlinear heat-conduction equation

$$\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}\left[k(T)\frac{\partial T}{\partial x}\right] \tag{1}$$

and compare its wave solutions with solutions of the linear hyperbolic equation

$$\frac{1}{g^2}\frac{\partial^2 T}{\partial t^2}+\frac{c_v\rho}{\lambda_0}\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial x^2}. \tag{2}$$

According to the data [1], $c_V\rho/\lambda_0 \approx 0.753\cdot 10^{-3}$ sec/cm^2 for helium at 2°K. In this case, (2) can be replaced by the following

$$\frac{\partial^2 T}{\partial t^2}=g^2\frac{\partial^2 T}{\partial x^2}, \tag{3}$$

which describes the propagation of undamped thermal waves. To find the wave solutions, we go over to the wave variable

$$\xi=v(x)+ct \tag{4}$$

in (1). Let us mention the transformation formula

$$\frac{\partial T}{\partial t}=c\frac{dT}{d\xi};\quad \frac{\partial T}{\partial x}=\frac{dv}{dx}\frac{dT}{d\xi};$$
$$\frac{\partial^2 T}{\partial x^2}=\frac{d^2v}{dx^2}\frac{dT}{d\xi}+\left(\frac{dv}{dx}\right)^2\frac{d^2T}{d\xi^2}.$$

Then taking into account that $d^2v/dx^2 = 0$, we will have

$$c-\frac{dk}{dT}\frac{dT}{d\xi}\left(\frac{dv}{dx}\right)^2=k(T)\frac{d^2T}{d\xi^2}\left(\frac{dv}{dx}\right)^2\bigg/\frac{dT}{d\xi}. \tag{5}$$

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For wave solutions to exist, (5) should be either algebraic, or an ordinary differential relative to $k(T)$ or $T(\xi, T)$.

Let $k = \alpha T^n$. Then we must set

$$\frac{dv}{dx} = b;\ T^{n-1}\frac{dT}{d\xi} = p;\ T^n\frac{d^2T}{d\xi^2} = q\frac{dT}{d\xi} \tag{6}$$

into (5), which yields, in turn

$$v = b_0 + bx;\ T = (np)^{\frac{1}{n}}\xi^{\frac{1}{n}};\ q = (1-n)p. \tag{7}$$

We hence obtain $c = \alpha pb^2$ from (5). The final solution for the direct wave has the form

$$T_1 = \left[\frac{nc}{ab^2}(b_0 + bx + ct)\right]^{\frac{1}{n}}. \tag{8}$$

The solution (8) was used in [2] for $b = -1$. We have an analogous formula for the reverse wave

$$T_2 = \left[\frac{-nc_0}{ab^2}(b_0 + bx - c_0t)\right]^{\frac{1}{n}}. \tag{9}$$

The presence here of the constant $b_0 \neq 0$ eliminates the singularity at the point $x = 0$, $t = 0$ for $n < 0$. Substituting (8) into (3) yields $c = gb$, from which the relationship between the relaxation parameter and the thermophysical constants (1) results. Since (3) has a solution in the form of (8) and (9), then (8) and (9) define the class of undamped wave solutions of (1).

In contrast to (1), Eq. (3) allows the superposition principle, in conformity with which a solution in the form of trigonometric functions can be obtained from the function $T = (np)^{1/n}\xi^{1/n}$. The presence of certain identical solutions for (1) and (3) permits the authors of [3] and [4] to reduce the problem of finding the thermophysical characteristics of the nonlinear heat-conduction equation to the problem of solving linear higher-order equations. Let us study the case $n = -1$. Let $|c| = |c_0| = c$, and let us form the difference $T = T_2 - T_1$. We will have

$$T = \frac{2ab^2t}{(b_0 + bx - ct)(b_0 + bx + ct)}.$$

Let us set $b_0 = 0$, $b = 1$, then

$$T = 2at/(x^2 - c^2t^2).$$

This formula is obtained in [5] in an analysis of self-similar solutions. It is a rare example of the superposition of two solutions for a nonlinear equation.

If $n = 1$, $b_0 = 0$, then we have from (8)

$$T_1 = \frac{c^2}{ab^2}\left(t - \frac{b}{c}x\right),$$

which is an example of the self-similar solution obtained in [6].

Let us examine still another example of a functional dependence of the thermal diffusivity coefficient $k(T) = A + BT$. Here (5) goes over into a nonlinear ordinary differential equation in $T(\xi)$

$$\frac{c}{b^2}\frac{dT}{d\xi} - B\left(\frac{dT}{d\xi}\right)^2 = (A + BT)\frac{d^2T}{d\xi^2}. \tag{10}$$

One of its particular solutions can be indicated

$$T = \frac{c}{Bb^2}\xi + p_0$$

that satisfies (3) also. It therefore also is in the class of undamped wave solutions.

2. Damped Thermal Waves

Let us set

$$\frac{dv}{dx} = b; \quad \frac{dT}{d\xi} = f_1(T); \quad \frac{d^2T}{d\xi^2} = f_2(T)\frac{dT}{d\xi}, \tag{11}$$

into (5), whereupon

$$f_1(T)\frac{dk}{dT} + f_2(T)k(T) = \frac{c}{b^2}. \tag{12}$$

If (11) and (12) are solved jointly, then we have its corresponding function k(T) for each function $T(\xi)$. For instance, let $f_1(T) = -\beta T$, $f_2(T) = -\beta$, then the integrals of (11) and (12) take the form

$$T = A\exp[-\beta(bx + ct)]; \quad k(T) = -\frac{c}{\beta b^2} + BT^{-1}. \tag{13}$$

It is easy to obtain the solution for the reverse wave also:

$$T = A\exp[-\beta(bx - c_0t)]; \quad k(T) = \frac{c_0}{\beta b^2} + BT^{-1}. \tag{14}$$

In turn, the hyperbolic equation (2) has two solutions

$$T_1 = A\exp\left[-\frac{g(x-gt)}{k_0 - \alpha g^2}\right]; \quad T_2 = A\exp\left[-\frac{g(x+gt)}{\alpha g^2 - k_0}\right]. \tag{15}$$

For (15) to go over into (13) and (14), it is necessary to set

$$g = \beta; \quad \frac{1}{\alpha g^2 - k_0} = b; \quad \frac{g}{\alpha g^2 - k_0} = c.$$

In problems of heat transfer to the soil, a heat-conduction model exists according to which the thermophysical parameters are functions of the coordinates and time. Empirical data on these functions are presented in [7] for the majority of soils of the Soviet Union. Also presented there are test data according to which the heat-conduction coefficient in the upper layer of the soil varies as a sinusoid during the day.

In conformity with this, (1) must be rewritten thus

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left[k(x,\ t)\frac{\partial T}{\partial x}\right]. \tag{16}$$

which respectively becomes, with (2), upon going over to the wave variable:

$$c - b^2\frac{dk}{d\xi} = \frac{b^2k\frac{d^2T}{d\xi^2}}{\frac{dT}{d\xi}}, \tag{17}$$

$$\frac{c_v\rho c}{\lambda_0} = \left(b^2 - \frac{c^2}{g^2}\right)\frac{\frac{d^2T}{d\xi^2}}{\frac{dT}{d\xi}}. \tag{18}$$

The $T(\xi)$ determined from the following ordinary differential equation

$$\frac{d^2T}{d\xi^2} = \text{const}\,\frac{dT}{d\xi}$$

will be the identical solution for (17) and (18). Then (17) is transformed into a differential equation for $k(\xi)$. However, there is an arbitrariness in the selection of the function $k(\xi)$, consequently, it is possible to proceed along the road of substituting empirical formulas for $k(\xi)$ in (17) and to determine $T(\xi)$ from the differential equation obtained here.

For instance, let $k(\xi) = \sin\xi$, then in place of (17) we will have

$$\frac{d^2T}{d\xi^2} = \frac{c/b^2 - \cos\xi}{\sin\xi}\,\frac{dT}{d\xi}.$$

Let us indicate the first integral of this equation

$$\frac{dT}{d\xi} = \ln\left(c_0\left|\operatorname{tg}\frac{\xi}{2}\right|^{\frac{c}{b^2}} \Big/ |\sin\xi|\right).$$

Because of the presence of a broad set of functions $k(\xi)$, Eq. (17) is richer than (18) in its solutions. However, the hyperbolic operator (2) can be extended by the introduction of higher derivatives with respect to time of any order [8, 9].

3. Material Constants of the Heat Field

The derivation of some formula for the heat-conduction equation is accomplished from the heat-balance equation

$$c_v\rho\,\frac{\partial T}{\partial t} = -\operatorname{div}\vec{q}. \tag{19}$$

This relationship is valid for every place and each instant, and from it there follows that the heat field is characterized by temperature and heat flux fields. The Fourier hypothesis

$$\vec{q} = -\lambda_0\operatorname{grad}T \tag{20}$$

permits calculating the heat flux field completely uniquely from the field of temperature gradients. At the very same time, this hypothesis introduces a scalar material constant λ_0, that characterizes the heat field of this substance and should possess at least some degree of universality.

If (20) is inadequate to the evaluation of the heat flux in terms of the temperature, then the question arises of whether the material constant λ_0 should be conserved upon further refinement of this relationship. If it is assumed that $\lambda_0 = \lambda_0(T)$ or $\lambda_0 = \lambda(x, y, z, t)$, then firstly, it is not known from what considerations to determine the mentioned functions, and secondly, the dimensionality of the constant appearing here will depend on the form of the mentioned functions. This latter circumstance does not permit characterization of the heat field by universal constants.

However, the presence of identical solutions of (1) and (2) indicates the legitimacy of such a method of refining (20), for which an additional material constant is introduced. It is here possible to propose a method of refining (20) for which the number of material constants will be arbitrary.

Indeed, the Fourier hypothesis asserts that the heat flux depends on the time and coordinates only in terms of the field of temperature gradients, i.e., implicitly. We assume the vector $\vec{q}$ to depend explicitly on the time t. Then (19) can be differentiated with respect to the time any number of times. We differentiate it twice in succession

$$c_v\rho\frac{\partial^2T}{\partial t^2} = -\frac{\partial}{\partial t}(\operatorname{div}\vec{q}); \tag{21}$$

$$c_v\rho\,\frac{\partial^3T}{\partial t^3} = -\frac{\partial^2}{\partial t^2}(\operatorname{div}\vec{q}). \tag{22}$$

We multiply (21) and (22) by the arbitrary constants α and β, respectively, and we then combine the equalities obtained with (19). We afterwards obtain

$$c_v\rho\left(\beta\frac{\partial^3 T}{\partial t^3}+\alpha\frac{\partial^2 T}{\partial t^2}+\frac{\partial T}{\partial t}\right)=-\operatorname{div}\vec{Q}. \quad (23)$$

The notation for $\vec{Q}$ is here

$$\vec{Q}=\vec{q}+\alpha\frac{\partial\vec{q}}{\partial t}+\beta\frac{\partial^2\vec{q}}{\partial t^2}. \quad (24)$$

Now, we express the vector $\vec{Q}$ in terms of the field of temperature gradients in conformity with the Fourier hypothesis

$$\vec{Q}=-\lambda_0\operatorname{grad}T \quad (25)$$

which permits rewriting (23) thus

$$\beta\frac{\partial^3 T}{\partial t^3}+\alpha\frac{\partial^2 T}{\partial t^2}+\frac{\partial T}{\partial t}=k_0\nabla^2 T. \quad (26)$$

Here two additional constants have been inserted to the standard thermal diffusivity constant k_0. Taking account of (25), Eq. (24) now becomes

$$\beta\frac{\partial^2\vec{q}}{\partial t^2}+\alpha\frac{\partial\vec{q}}{\partial t}+\vec{q}=-\lambda_0\operatorname{grad}T. \quad (27)$$

A more general form of this equation is obtained in [8] by using finite-difference representations.

4. Initial Conditions

The introduction of higher derivatives into the heat-conduction equation requires reexamination of the initial and boundary conditions. This question is formulated most completely in [10]. However, an example of an initial condition can be indicated such that solutions of the parabolic heat-conduction equation cannot satisfy.

Let us consider the telegraph equation

$$\alpha\frac{\partial^2 T}{\partial t^2}+\frac{\partial T}{\partial t}=k_0\nabla^2 T. \quad (28)$$

Let the temperature be a periodic function of the coordinates, i.e.,

$$T=f(x,\ y,\ z)\,\theta(t),\quad \nabla^2 T=-a^2 f(x,\ y,\ z)\,\theta(t).$$

For such periodic functions (28) goes over into an ordinary differential equation that agrees with the formula for the equation of oscillations of a material point

$$\frac{\alpha}{a^2}\,\frac{d^2\theta}{dt^2}+\frac{1}{a^2}\,\frac{d\theta}{dt}+k_0\theta=0. \quad (29)$$

If $4\alpha k_0 a^2 \ll 1$, then its solution has the form

$$\theta=A_1\exp\left[-a^2k_0t\right]+A_2\exp\left[-\frac{1}{\alpha}\,t\right].$$

For this equality to satisfy the zero initial condition, it must be rewritten thus

$$\theta=A\left(\exp\left[-a^2k_0t\right]-\exp\left[-\frac{1}{\alpha}\,t\right]\right). \quad (30)$$

As $\alpha \to 0$ and $t \neq 0$ the complete solution of (30) goes over into the solution of the simplified differential equation corresponding to the parabolic heat-conduction operator. The solution mentioned has the form

$$\theta = A \exp [a^2 k_0 t]. \tag{31}$$

Both solutions agree for sufficiently large values of t, but they will behave differently at the initial instant. Therefore, for a more exact description of the temperature field at any time, the hyperbolic equation (28) must be used in this case. The utilization of (26) can extend the class of initial conditions still more.

The above holds in the theory of a viscous fluid, to which Prandtl turned attention (see [11]). There (31) and (32) correspond to solutions of the Euler and Navier–Stokes equations, and the zero initial condition is equivalent to the condition of fluid adhesion to a wall.

At an earlier stage of using (28) in the theory of heat conduction, the material constant α was identified with the Maxwell relaxation time [12], which is a small quantity. This latter sharply constrained the domain of application of the equation mentioned. If (28) is supplemented by a relation for the heat flux

$$\alpha \frac{\partial \vec{q}}{\partial t} + \vec{q} = -\lambda_0 \operatorname{grad} T, \tag{32}$$

then the parameter α can be determined from the law of variation of $\vec{q}$ on the body boundary.

Indeed, let $T = e^{-\gamma t} u(x, y, z, t)$, then we will have

$$\frac{1}{g^2} \frac{\partial^2 u}{\partial t^2} = \nabla^2 u + \frac{\gamma}{2k_0} u \tag{33}$$

in place of (28). Here taken additionally into account is

$$1 - \frac{2\gamma}{g^2} = 0, \ \alpha = \frac{1}{2\gamma}. \tag{34}$$

On the body boundary the relationship (32) becomes

$$\frac{1}{2\gamma} \frac{d\vec{q}}{dt} + \vec{q} = -\lambda_0 e^{-\gamma t} \operatorname{grad} u|_{x=y=z=0}. \tag{35}$$

Therefore, the vibrational nature of the heat flux on the body boundary can cause temperature waves within the body, whose attenuation will be determined by the material constants λ_0 and k_0.

A structural analysis of the heat conductivity [13], which permits the study of the influence of heat flux on the body boundary on the process of heat conductivity in the whole domain by simple means, is especially effective in a joint study of (26) and (27).

NOTATION

$k(T)$, thermal diffusivity coefficient; α, relaxation parameter; $g^2 = \lambda_0/c_V\rho\alpha$, square of the heat wave velocity; λ_0 and k_0, heat-conduction and thermal diffusivity constant; ρ, density; c_V, specific heat at constant volume; T, temperature; t, time; x, y, z, space coordinates.

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